Pipe Network Analysis:
An Exercise in Computer-Aided Analysis

David R. Sawyers, Jr.
Ohio Northern University, Ada, OH 45810
Email: d-sawyers@onu.edu

Abstract

A problem from fluid mechanics is used in an engineering analysis course to expose senior mechanical engineering students to several important issues related to the use of sophisticated general-purpose analysis software. The problem, which models incompressible flow in a pipe network with both parallel and series branches, results in a very large system of coupled, nonlinear algebraic equations (the formulation provided here results in seventy variables). Once a well-posed mathematical problem has been specified, a numerical solution is obtained using a general-purpose nonlinear equation solver called EES. However, obtaining converged, reasonable solutions to the specified problem requires a significant amount of care, both in setting up the equations and in controlling the solver.

Introduction

Numerical analysis has become a common tool for solving engineering problems[1,2]. As a result, today's undergraduate student is able to quickly solve problems that would have required significant expenditures of time and resources a generation ago. However, the availability of user-friendly, high-level solution packages can also give students a misguided sense of confidence in their numerical solutions to complex problems. This paper describes an assignment requiring students to develop and solve a large system of coupled, nonlinear algebraic equations governing flow in a piping network. The problem offers several important lessons about numerical methods. First, it was necessary to formulate the governing equations carefully to avoid unnecessary complexity. In addition, achieving convergence required judicious selection of initial guesses and careful control of the iteration process. Finally, several groups obtained result which (for a variety of reasons) were not correct, but were not blatantly wrong - emphasizing the essential role of validation in the numerical process. After completion of the assignment, students were asked to identify the primary obstacle encountered in solving the problem. Their responses suggest that the assignment led most students to an increased appreciation for the complexities of numerical methods.

Course Description

ME-4511 Engineering Analysis is taken by mechanical engineering students during the fall semester of their final year. The course is intended to serve as a one-semester "analysis capstone" course to complement the year-long design capstone that has long been required for mechanical engineering students at Ohio Northern University. The course also serves as a partial replacement for two courses (previously taught under the quarter system): an engineering problem-solving course taught in the fall of the junior year, and a numerical methods course taught in the spring of the junior year.

In addition to lectures and homework on a variety of analytical and numerical problem-solving tools, students are assigned several "extended problems." These extended problems are intended to draw upon knowledge gained both in ME-4511 and in previous courses. All of these extended problems require application of prior knowledge to more complex problems, while some also require solution of ambiguously-defined problems; some of the problems cross the boundaries of traditional mechanical engineering courses.

The course was taught for the first time during the fall semester of 2012, with an enrollment of 20 students. Students are allowed to work in teams of two to solve the extended problems, although on the particular problem discussed here there were two individual submissions. The course outcomes stated on the syllabus include the following:

Upon completion of the course, students will be able to:

1. *solve engineering problems using a variety of analysis methods and software tools.*
2. *apply numerical techniques such as Runge-Kutta methods and finite-difference methods to obtain solutions to differential equations that apply to engineering practice.*
3. *solve problems which are not well-defined, or do not have an obvious closed-form solution.*
4. *solve problems which cross the traditional boundaries of mechanical engineering courses.*

The assignment described in this paper addresses outcome 1 by requiring students to solve a large set of coupled, nonlinear algebraic equations using the software package EES (Engineering Equation Solver). The assignment also addresses outcome 3 in that, although the underlying physics is well-defined, the problem can be formulated several ways, some of which are more readily amenable to numerical solution than others.

Problem Assignment

The problem statement was presented to the students in the following form:

Consider the fluid pipe network shown below, with one inlet at a pressure P_1= 60 psig = 413,700 Pa and four outlets (at points 4, 6, 8, and 10), all at atmospheric pressure. All pipes have a diameter of 0.1 m and are made of cast iron. The lengths of each section are L_A = 5m, $L_B = L_D = L_F$ = 3m, $L_C = L_E = L_H = L_J$ = 2m, and $L_G = L_I$ = 5m. There are valves at each outlet with loss coefficients $K_{LC} = K_{LE} = K_{LH} = K_{LJ}$ = 0.2 (although these should be adjustable). The working fluid is water. All pipe sections are at the same elevation. Neglect minor losses associated with fittings other than the valves.

a) Determine the volumetric flowrate at each exit.
b) Set the flowrates at the exits to QC = QE = QH = QJ = 0.05 and determine the required valve settings (K_{LC}, K_{LE}, K_{LH}, K_{LJ})

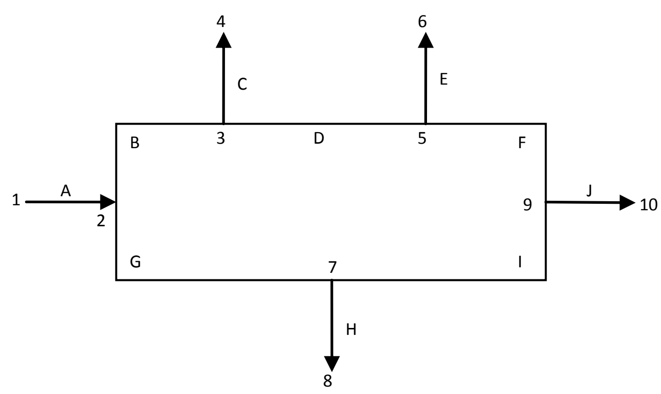

Figure 1: Problem statement for the Pipe Network extended analysis problem. Although flow direction is suggested by the arrows, it may be possible (for part a) to obtain negative flowrates.

Each team was required to submit a memo report describing the appropriate set of equations required to define a well-posed mathematical problem, along with a description of the solution procedure. It was recommended that the problem be solved using EES (Engineering Equation Solver), a coupled nonlinear equation solver that students had experience with in several prior engineering courses.

The governing equations and supporting information needed to solve the problem were covered in a prerequisite course, ME-3311 Fluid Mechanics. However, while pipe flow in parallel and series was discussed in that course, the current problem is much larger in scale. As will be discussed in the next section, the problem (as formulated by the author) consists of ten governing equations and twenty supporting equations, plus a number of variable definitions, resulting in a large system of coupled, nonlinear equations.

In their previous experience with EES, students were able to solve a wide range of problems in thermodynamics, fluid mechanics, machine design, and dynamic systems without much regard for underlying numerical issues such as initial guesses and convergence. However, due to the complexity of the problem described in Figure 1, obtaining an accurate solution required a great deal of care both in reformulating the mathematical problem as an EES problem and in controlling the iterative solution procedure.

Solution

If we assume the water behaves as an incompressible fluid, applying the steady-state conservation of mass equation at node 1 gives:

$$Q_A = Q_B + Q_G$$

where Q_i is the volumetric flowrate (in m^3/s) in pipe segment i, with subscripts corresponding to the segment lettering shown in Figure 1. Since the diameter of all sections is constant, we can use velocities instead of flowrates. Although this may seem trivial, it actually reduces the complexity of the numerical problem by reducing the number of primary unknowns (since flowrate becomes a secondary value obtained from the velocity). Applying conservation of mass at each node results in five equations for the unknown velocities V_i:

$$V_A = V_B + V_G \tag{1a}$$

$$V_B = V_C + V_D \tag{1b}$$

$$V_D = V_E + V_F \tag{1c}$$

$$V_G = V_H + V_I \tag{1d}$$

$$V_J = V_F + V_I \tag{1e}$$

The principle of conservation of energy for pipe flow problems is usually expressed as the head loss equation. Applying the head loss equation between the inlet at state 1, and the exit at state 4 gives:

$$\left(\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1\right) + h_p = \left(\frac{P_4}{\rho g} + \frac{V_4^2}{2g} + z_4\right) + h_t + h_L$$

where P_j is the pressure at inlet or exit j (in Pa), ρ is the fluid density (in kg/m^3), g = 9.81 m/s^2 is the gravitational constant, V_j is the average velocity at the inlet/exit, z_j is the elevation of the inlet/exit, and h_L is the head loss due to friction. The subscripts correspond to the pipe junction numbers shown in Figure 1. Since the inlet pressure is specified as a gage pressure and the exits are all at atmospheric pressure, the exit pressure is zero. Note that $V_1 = V_A$, $V_4 = V_C$, etc. Since there are no elevation changes, the head loss equation reduces to:

$$V_C^2 = 2g\left(\frac{P_1}{\rho g} + \frac{V_A^2}{2g} - h_{LA} - h_{LB} - h_{LC}\right)$$

with

$$h_{LA} = \frac{f_A L_A}{D}\frac{V_A^2}{2g} \qquad h_{LB} = \frac{f_B L_B}{D}\frac{V_B^2}{2g} \qquad h_{LC} = \left(\frac{f_C L_C}{D} + K_{LC}\right)\frac{V_C^2}{2g}$$

where f_i is the friction factor in pipe segment $_i$, L_i is the length of the pipe segment, D is the diameter, V_i is the average velocity within the pipe segment, and g is the gravitational constant. The terms h_{LA} and h_{LB} correspond to the major head losses due to friction in the corresponding pipe section. The term h_{LC} includes both the major losses and the minor loss due to a valve with loss coefficient K_{LC}.

Applying the head loss equation between state 1 and each exit (including separate equations for both the upper loop and the lower loop for states 1→10) results in more five equations for the unknown exit velocities:

$$V_C^2 = 2g\left(\frac{P_1}{\rho g} + \frac{V_A^2}{2g} - h_{LA} - h_{LB} - h_{LC}\right) \qquad (2a)$$

$$V_E^2 = 2g\left(\frac{P_1}{\rho g} + \frac{V_A^2}{2g} - h_{LA} - h_{LB} - h_{LD} - h_{LE}\right) \qquad (2b)$$

$$V_J^2 = 2g\left(\frac{P_1}{\rho g} + \frac{V_A^2}{2g} - h_{LA} - h_{LB} - h_{LD} - h_{LF} - h_{LJ}\right) \qquad (2c)$$

$$V_H^2 = 2g\left(\frac{P_1}{\rho g} + \frac{V_A^2}{2g} - h_{LA} - h_{LG} - h_{LH}\right) \qquad (2d)$$

$$V_J^2 = 2g\left(\frac{P_1}{\rho g} + \frac{V_A^2}{2g} - h_{LA} - h_{LG} - h_{LI} - h_{LJ}\right) \qquad (2e)$$

Although these equations could be solved for V_i, taking the square root of the right-hand side can lead EES to fail to converge if these terms becomes negative during iteration. While it would be possible to write the argument of the square root as a separate variable and limit it to non-negative values, this is not necessary since the formulation shown in equations 2a-e provides a convergent solution.

Equations 2a-e have introduced ten new unknown head loss terms (h_{Li}). The definition of the head loss due to friction provides ten additional equations:

$$h_{LA} = \frac{f_A L_A}{D}\frac{V_A^2}{2g} \quad (3a)$$

$$h_{LB} = \frac{f_B L_B}{D}\frac{V_B^2}{2g} \quad (3b)$$

$$h_{LC} = \left(\frac{f_C L_C}{D} + K_{LC}\right)\frac{V_C^2}{2g} \quad (3c)$$

$$h_{LD} = \frac{f_D L_D}{D}\frac{V_D^2}{2g} \quad (3d)$$

$$h_{LE} = \left(\frac{f_E L_E}{D} + K_{LE}\right)\frac{V_E^2}{2g} \quad (3e)$$

$$h_{LF} = \frac{f_F L_F}{D}\frac{V_F^2}{2g} \quad (3f)$$

$$h_{LG} = \frac{f_G L_G}{D}\frac{V_G^2}{2g} \quad (3g)$$

$$h_{LH} = \left(\frac{f_H L_H}{D} + K_{LH}\right)\frac{V_H^2}{2g} \quad (3h)$$

$$h_{LI} = \frac{f_I L_I}{D}\frac{V_I^2}{2g} \quad (3i)$$

$$h_{LH} = \left(\frac{f_J L_J}{D} + K_{LJ}\right)\frac{V_J^2}{2g} \quad (3j)$$

These equations have in turn added ten unknown friction factors f_i. The standard approach for calculating the friction factor f for turbulent flow in pipes is to use the Moody chart (which is not useful for a numerical solution) or the Colebrook equation[3]:

$$\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{\varepsilon}{3.7D} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right)$$

where ε is the roughness of the pipe (a property of the pipe material), and Re is the Reynolds number (defined below). Unfortunately, this equation is implicit in f, which causes numerical problems when implementing it in our EES solution. An alternative is the Haaland equation, which is reported to provide friction factor values within 2% of the Colebrook equation[3]:

$$\frac{1}{\sqrt{f}} = -1.8\log\left[\frac{6.9}{\mathrm{Re}} + \left(\frac{\varepsilon}{3.7D}\right)^{1.11}\right]$$

This can be solved explicitly for f:

$$f = -1.8\log\left[\frac{6.9}{\text{Re}} + \left(\frac{\varepsilon}{3.7D}\right)^{1.11}\right]^{-2}$$

Implementing this equation directly into EES causes problems because the argument of the logarithm can become negative during iteration, even though the converged value is positive. In addition, during iteration the negative exponent caused an error due to division by zero. These numerical difficulties can be overcome by separating the Haaland equation into three separate variables. As a result, the equations for the friction factor f_i in pipe segment i have the following form:

$$fff_i = \frac{6.9}{\text{Re}_i} + \left(\frac{\varepsilon}{3.7D}\right)^{1.11} \qquad \text{(4a-j)}$$

$$ff_i = -1.8\log\left[fff_i\right] \qquad \text{(5a-j)}$$

$$f_i = 1/\,ff_i^2 \qquad \text{(6a-j)}$$

This formulation allows the iterative values of these variables to be restricted to non-negative numbers through the EES “Variable Info” window (Figure 2).

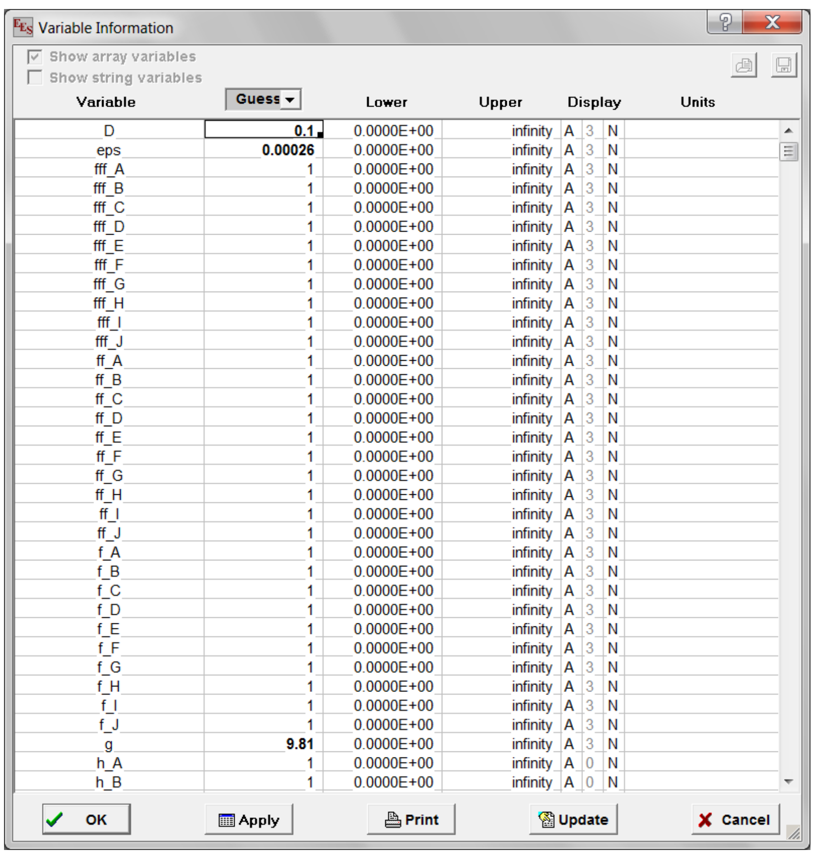

Figure 2: The EES Variable Info window.

The Reynolds number for each pipe segment is defined as:

$$\mathrm{Re}_i = \frac{\rho V_i D}{\mu} \qquad (7a\text{-}j)$$

Finally, the numerical model is completed by including the relationship between velocity and flowrate:

$$Q_i = \frac{\pi V_i D^2}{4} \qquad (8a\text{-}j)$$

Equations 1-8 define the set of seventy coupled equations to be solved in EES.

Obstacles to Solution

Most student teams were able to produce a properly developed mathematical model with little or no assistance from the instructor, instead relying primarily on knowledge gained in the prerequisite fluid mechanics class (and the corresponding textbook). However, all of the students required some help in obtaining the correct numerical solution. Due to a change in academic calendars (transitioning from quarters to semesters), enrollment in the fall 2012 semester was smaller than usual. As a result, the instructor was able to review the solution of each student group in depth. The following is a summary of significant issues encountered by the students. This list does not include typos and other programming errors.

> *Issue 1: Applying conservation of mass to the overall system.* As described in the development of equation 2 above, applying the head loss equation (conservation of energy) between the inlet and each exit results in four equations. An additional equation is required, which can be obtained by applying the head loss equation between the inlet and one of the exits twice, with each equation representing a different flow path (equations 2c and 2e). However, several student groups obtained the required additional equation by applying conservation of mass to the overall system ($Q_{inlet} = \Sigma Q_{exits}$), which does not lead to a convergent solution. In fact, the overall conservation of mass equation is simply a linear combination of equations 1a-e, so this formulation does not result in a well-posed problem.
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> *Issue 2*: *Use of the Colebrook equation for friction factor leads to an EES program that does not converge.* The Colebrook equation is listed in many standard Fluids texts as the primary equation for calculating friction factor, which led several groups to use this approach. However, the implicit nature of the Colebrook equation results in a system of equations that seems to be beyond the capability of the EES solver.
>
> *Issue 3*: *Implementation of the Haaland equation for friction factor as a single equation leads to a program that does not converge.* As described in the development of equations 4-6 above, the iterative nature of the EES solver leads to a program that invariably either attempts to calculate the logarithm or the inverse of a negative number. Since this issue obviously does not occur for the converged vales, the problem may not occur for some sets of initial guesses. However, every student group that used a single equation for the

Haaland equation encountered a convergence error, suggesting that choosing a set of initial guesses that avoids these problems is extremely difficult.

Issue 4: *Choosing an appropriate set of initial guesses is necessary for convergence.* As an iterative solver, EES begins with a set of initial guesses for each variable. The default value is 1, but this can be modified by the user. The default value is sufficient for most variables, but it is necessary to set the Reynolds numbers to a relatively large value (e.g. 1000) in order to converge. The converged values of Reynolds number in each pipe section are on the order of 10^5 - 10^6, so it is not necessary for the initial guess to closely match the converged values.

Issue 5: *Setting inappropriate convergence criteria leads to solutions which have not converged, yet may not be obviously incorrect.* After eliminating issues 1-4, several groups obtained convergent solutions which were not correct, but were of the same order of magnitude as the correct solution. In some cases this was due to typing errors, but in other cases the convergence criteria had been eased by the students - probably while trying to fix divergence due to one of the previous issues.

The significant issues described above relate either to developing a well-posed problem (Issue 1), writing the problem in a format that allows control over the iterative process (Issues 2-4), or properly understanding convergence of iterative analysis methods (Issue 5).

Conclusion

In addition to the explicit outcomes stated in the introduction, the author hoped to accomplish two additional informal objectives with this assignment. The first objective was to develop a sense of healthy skepticism when using numerical methods to solve engineering problems. The availability of powerful and user-friendly software such as EES, while valuable to the practicing engineer, can also appear to provide deceptively simple solutions to complex problems. The second objective was to develop each student's confidence in his or her ability to solve complex engineering problems. In most undergraduate courses, the students appear to rely heavily on "solution by example." When a problem does not closely match an example that has been seen before, there is a tendency in some students to assume that a solution is beyond their ability. All of the extended problems in ME-4511 Engineering Analysis address this issue to some extent by requiring students to solve problems of greater complexity than those found in the typical introductory engineering textbooks. The problem presented in this paper does so by increasing the scale of the problem far beyond that which can be solved by hand.

This assignment was successful at accomplishing the first of these two objectives. One lesson clearly learned by every group is that a seemingly simple-to-use analysis tool such as EES can have a great deal of complexity that is hidden from the casual user. Most of the students had never encountered a convergence error in their previous use of EES, and most did not clearly understand the iterative approach used by the solver. Understanding this iterative approach becomes critical when a solution is strongly dependent on initial conditions, or when it is necessary to restrict intermediate values to a certain range (e.g. avoiding non-negative numbers while iterating).

From a practical standpoint, this assignment was effective at teaching students some of the complexities of numerical methods. Every group was required to produce an EES model that produced the correct solution, even though for some students this required several attempts and a significant amount of assistance from the instructor. The result was that, rather than giving up after their first attempt, they were forced to identify (or at least acknowledge) and correct the problems in their model. When asked to identify the major obstacles they encountered in obtaining the correct solution, students provided a range of responses that correspond to some extent to the issues described in the previous section. Table 1 below shows a summary classification of the responses.

Table 1: Student identification of significant obstacles to

obtaining the correct solution.

Obstacle (Paraphrased)	**Number of times mentioned**
Need to set appropriate initial guesses	12
Writing governing equations in a suitable form	7
Need to set limits on variable range during iteration	4
Identifying an incorrect solution	1
Obtaining convergence	1

It is more difficult to conclude whether the second objective - increasing the students' confidence in their ability to solve complex problems - was achieved to a significant extent. On the one hand, all of the groups were required to obtain the correct solution. However, several groups required a significant amount of assistance from the instructor. In addition, a degree of frustration was clearly indicated in many of the responses to the survey conducted after the problem was completed. Some thought may be needed to develop strategies to reduce this frustration to the point where it doesn't overwhelm any sense of accomplishment the students might feel. One such strategy that will be implemented is to require students to present their mathematical model (the equations to be entered into EES) as a preliminary assignment. This should increase the distinction between modeling issues and numerical problems. A second strategy will be to spend more class time discussing iterative numerical methods, including the importance of initial guesses and the need to limit variables to certain ranges to avoid division by zero or similar mathematical problems.

Acknowledgement
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